

# Well ARMed and FiRM: diversification of mortgage loans for homebuyers

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*Individual homebuyers today are offered a wide range of mortgage options for financing the purchase of a house. Usually, homebuyers are also granted an option to repay the mortgage loan and in some countries (such as Denmark) it is particularly efficient to do so as market conditions change or the homebuyer's situation warrants it. While, traditionally, a single mortgage loan would serve borrower needs, today it appears that a portfolio of loans may satisfy the mortgage needs of the individual and their appetite for risk much better. In this paper we develop a model for the diversification of mortgage loans of a homebuyer and apply it to data from the Danish market. Even in the presence of mortgage origination costs, it is shown that most risk-averse homebuyers will do well to consider a diversified portfolio of both fixed- and adjustable-rate mortgages (FRM and ARM). This is particularly the case if one takes, unavoidably, a long-term perspective in financing the purchase of a home through a mortgage loan.*

## 1 INTRODUCTION

What is in a name? Everything is in the name, the marketing guru will tell us, and the names used to pitch the hottest subset of the old-fashioned mortgages tell a lot: interest-only (IO), Option ARM and Pick-a-Payment have been added to the traditional shopping list of fixed- or adjustable-rate mortgages (FRM and ARM).

In essence all of these mortgage products aim at satisfying the same underlying need of borrowers: offering them a loan to achieve their immediate home-purchasing goals, with payment terms that can be adapted as the family earnings change (usually with an upward adjustment, especially for young homebuyers), while offering some protection from market changes to both borrowers and lenders. The issue, especially for first-time buyers, has been one of trading-off the lower rate and payment on an ARM with the interest rate risk of the

ARM, or going for the higher initial payments of a FRM and lower risk when the FRM is kept to maturity. With an IO mortgage the initial payments can be reduced substantially, but future payments will increase significantly when the IO clause expires and principal payments must be made, especially if there have been significant changes in interest rates since the loan was issued.

According to some surveys (Real Estate Center (2002)) innovative mortgage products account for half of the new mortgages written in the US, up from less than 10% in 2001. The innovations in the Danish market have not been as exotic as their US counterparts, but they have been particularly simple and efficient to promote. The seven specialized mortgage banks that operate in Denmark fund the loans by issuing bonds in the capital markets. The terms of the bonds are identical to the mortgage loan they fund, with the mortgage bank adding a markup on the market yield of bonds with maturity comparable to the loan. Given the readily available information of market yields, all banks offer the same markup rate, and this mechanism is equivalent to the borrower issuing their own bonds. Furthermore, all borrowers pay the same rate on the same type of loan issued on a given date. (This is possible, even accounting for differences in credit worthiness of the borrowers, as most loans require a 20% down payment which adequately covers a wide range of credit-risky borrowers. However, borrowers with very poor credit or without initial endowment are not served in this market.)

Given the simplicity with which market rates are transformed into mortgage loans, on any given day a borrower can be offered a range of FRM with 15-, 20- or 30-year maturity and ARM with adjustment periods ranging from one to 10 years and these products reach a wide market segment. The total amount of mortgage loans issued in Denmark (this includes the origination of new loans plus the refinancing of old loans) in 2005 totals almost €100 billion. This represents a record increase of 25% from the year before and corresponds to half of the Danish gross domestic product. FRM and ARM account for 50% of the mortgage market each. Innovative products such as IO and ARM with a cap are quite popular. About 30% of the total ARM in the market come with a cap, while about 30% of the total amount of mortgage loans in the private and summer house market are IO.

The refinancing activity in this market is also noteworthy. About one-third of all outstanding loans, for a total of €70 billion, were refinanced in 2005. The issue of new loans reached €20 billion, corresponding to a growth of 10% on the total amount of mortgage loans in the Danish market, which sets a new record.

Some form of protection from either market changes or changes in family conditions (job loss, births, deaths or divorce) comes in the form of an early prepayment option. Additional protection from interest-rate risk is offered through caps on the rate adjustment of ARM or with the purchase of FRM. However, then one has to deal with the higher rates associated with a FRM and accept the risk that rates will decline while payments on the loan remain locked. Furthermore, in the Danish market, early termination of a FRM requires prepayment of the mortgage at prevailing market prices which represents significant interest-rate risk for homebuyers with issued loans with prices below par, except those few who keep the mortgage to maturity.

## 2 ARE THERE DIVERSIFICATION BENEFITS FROM PORTFOLIOS OF MORTGAGE LOANS?

In the context introduced above it is somewhat surprising that the question of the diversification of homebuyer mortgage loans has not received any attention to date. This is primarily due to the transaction costs involved in obtaining more than one mortgage loan: mortgage origination fees in Denmark stand at €300 currently. However, in the efficient Danish market there has still been an interest in selling more than one product to each homebuyer. Indeed, it can be easily observed that a combination of mortgage loans could provide lower average total payments during the life of a mortgage and with less variability as interest rates change. Figure 1 illustrates the changes in total payment for FRM and an ARM with increasing interest rates. It is seen from this figure that the sensitivity of FRM and ARM to interest-rate changes have opposite signs, and hence a properly balanced portfolio of both types of loans could provide better protection than either mortgage alone.

The same conclusion is further highlighted from an analysis of the historical performance of mortgage loans in the Danish market during the period 1995–2005. Figure 2 shows the mean payment and the risk of the payment for various typical mortgages during this period. Risk is measured by the conditional value-at-risk (CVaR) at the 90% confidence level, see, Rockafellar and Uryasev (2000) or Jobst and Zenios (2001). The differences in mean payments and CVaR, coupled with the negative correlation of the FRM and ARM payments, suggest that interest-rate risk can be diversified by holding portfolios of mortgages.

The structuring of diversified portfolios is the topic of this paper. First, however, some more explanations are in order for the operations of the Danish mortgage market.

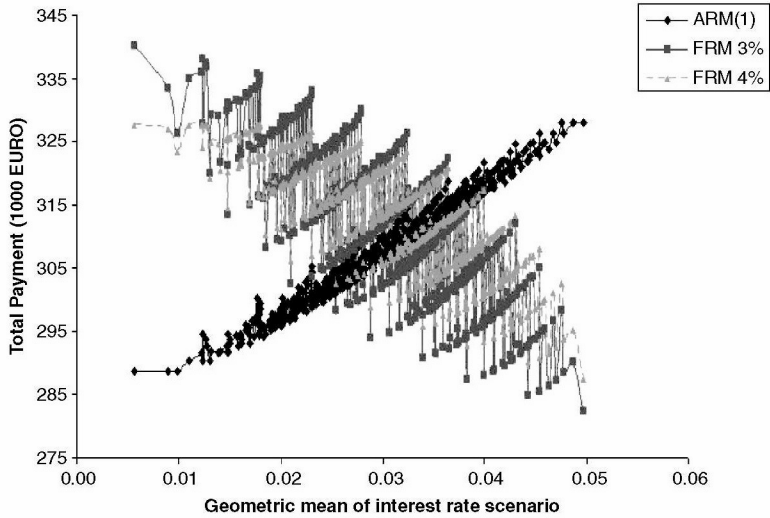
## 3 SOME EXPLANATIONS OF DANISH MORTGAGES

The Danish mortgage banks are highly specialized institutions whose line of business is, first, to collect the investments from the investors of mortgage backed securities and, second, to pool the investments together and issue mortgage loans to homebuyers. The great volume of housing trade (the outstanding mortgage debt corresponds to one-half of the gross domestic product of the country), the efficiency of the one-stop-shop process of mortgage origination by specialized banks, together with a 200-year history of no default from the mortgage banks<sup>1</sup> result in cheap loans for prospective house buyers. As investors are not exposed to default risk the Danish mortgage-backed securities are rated AAA and banks simply add to the market bond yields a markup rate that currently stands at 0.55%.

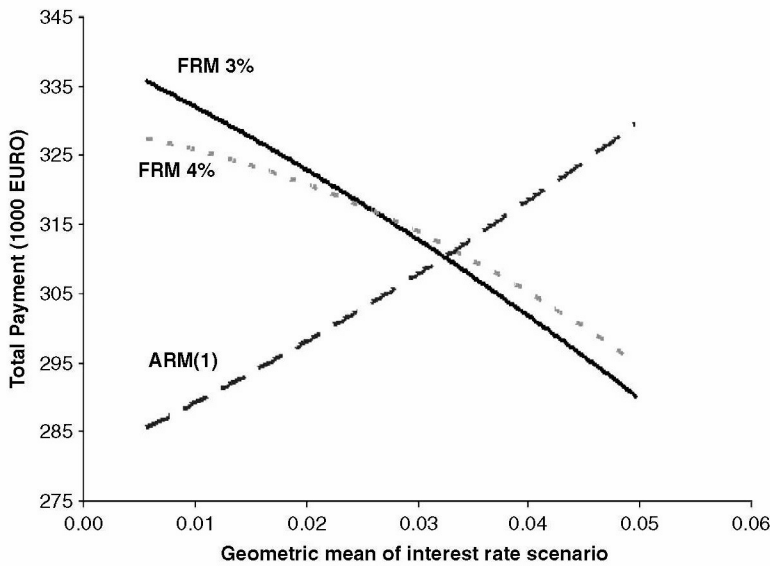
A unique feature of the Danish mortgage market is the “balance principle” prescribing that the payments made by the mortgagor are exactly the payments received by the investor. In effect, Danish mortgagors are trading mortgage bonds directly and may exercise all of the options. A Danish FRM has a call option

<sup>1</sup>Homebuyers may fail to pay their liabilities, but there has not yet been an incident of default when it comes to payments to investors via the mortgage banks.

**FIGURE 1** The total expected payment of 30-year FRM (3% and 4% FRM) and ARM(1) with annually adjusted rates have opposite sensitivities to changing interest rates: (a) all simulated data for a six-year planning horizon; (b) smoothed data using second-order approximations.

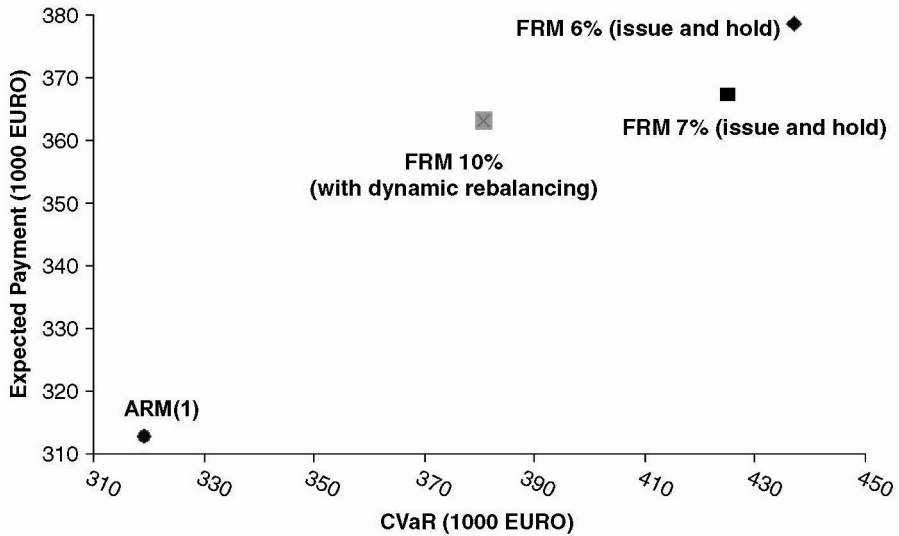


(a)



(b)

**FIGURE 2** Expected payment and risk of payments (measured by CVaR) for FRM and ARM in the Danish mortgage market during the period 1995–2005.



typically with strike price at 100 and a buy-back delivery option embedded in the underlying bonds. In particular, this has an impact on loans with long maturities, as small movements in interest rates result in big movements in the prices of FRM and, hence, have a direct impact on the amount of outstanding debt for the Danish mortgagor.

Until 1995, FRM were the only kind of mortgage-backed securities that were traded in the Danish market. Since then, the mortgage market has been growing fast and a number of new products have been introduced. The two most popular products have been the ARM loans with varying adjustment intervals and capped ARM loans where the interest rate cannot grow higher than a predetermined level.

All of these loans can be issued with IO payments for a grace period of up to 10 years, although after 10 years it is possible to refinance the loan with another 10 years of IO payments and as a result the outstanding debt is not being reduced during this period.<sup>2</sup>

The features of either FRM or ARM, together with the flexibility for refinancing the loans, imply that the total payments on the mortgage during the life of the loan is highly uncertain. While a mortgage owner will finance a loan in ways that are consistent with their short-term financial capabilities, in the long run the payments made and the outstanding debt will be determined by the changing interest rates. A simulation model can be used to project the total payments,

<sup>2</sup>This practice has been challenged in Real Estate Center (2002), where the question of whether rolling debt over long horizons could be a ticking time bomb for the mortgage market is raised.

including the value of the outstanding debt if the loan is refinanced before maturity; see Mulvey and Zenios (1994) on the use of simulation models for capturing correlations of fixed income securities. Payment projections are made based on an underlying interest-rate process (we use a variation of the Vasicek model in our work; see Appendix A) for all types of loans. The same process is also used to estimate the mortgage security market price for the outstanding debt and to determine the exercise of any options. The result is a distribution of net payments for different types of loans that can then be used to combine loans and obtain an optimized, diversified, portfolio. The simulation results for FRM and ARM given next further highlight the potential diversification effects of optimized portfolios.

### 3.1 Fixed-rate mortgages

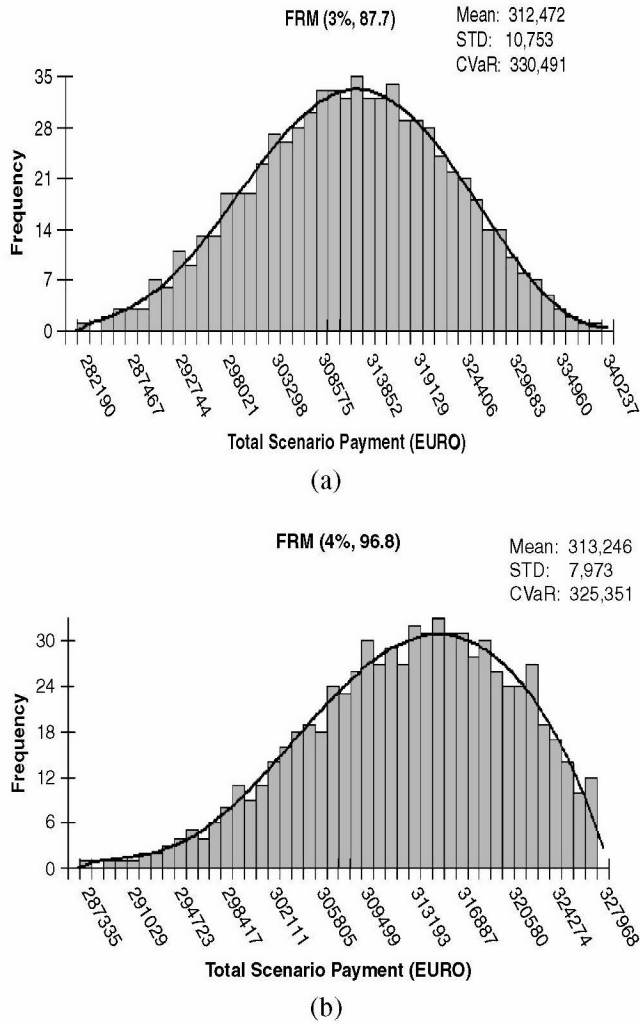
A FRM pays a fixed annual rate for the duration of the loan, normally 15, 20 or 30 years. In addition to the fixed rate there is also a price associated with a FRM, which is based on the amount paid by the investor to the mortgage bank upon loan origination. In particular, the interest rate and principal payment calculations are not based on the amount paid by the investor, but on the face value of the FRM. For example, if the price of a FRM is 96.8 then for every €96.8 that the mortgage receiver receives he will owe the investor €100.

As a result, although the interest-rate payments on a FRM are fixed the overall payment is not constant due to the fact that the price of a FRM changes as the general level of interest rates change and the Danish mortgage receiver has a buy-back delivery option on the FRM, which means that the mortgage can be prepaid at any time at the prevailing market price. Hence, unless the loan is kept until maturity (an unlikely situation), the borrower does not know with certainty the overall payments. This situation is illustrated using simulations in Figure 3(a), for a 30-year 3% FRM, which is prepaid after six years.

It is worth pointing out that, to the best of our knowledge, no mortgage banks outside Denmark offer this buy-back delivery option. In all other cases we are aware of, should the mortgage receiver wish to prepay the mortgage loan then payment is due of the original loan or any remaining part thereof. Most mortgage banks across the world of course offer a call option, so the mortgage receiver may prepay the mortgage at a predetermined price, usually par. The buy-back delivery (call) option of the Danish mortgages introduces an asymmetry in the payment distribution which is illustrated in Figure 3(b) for a 30-year 4% FRM with a price close to strike, when it is also prepaid after six years under different interest-rate scenarios.

Comparing the payments of the two FRM we observe that the FRM with the price closest to par has a smaller volatility of payments but a higher mean. Increased volatility is the price to pay for the upper bound on the payment. Some mortgage receivers are willing to pay higher mean payments in order not to worry about very high payments that might occur if the initial price of the FRM is below par and interest rates drop. Others may prefer the low original payment today to get into a new home, in expectation of higher income in the future. As a rule of thumb house buyers are not advised to issue FRM with prices below 95.

**FIGURE 3** Distribution of total payments, including interest payments and principal prepayment after six years, of 30-year FRM with different rates: (a) 3% and (b) 4%.

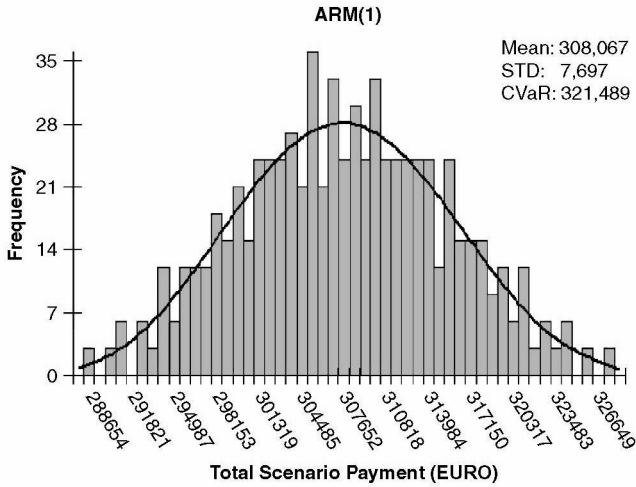


### 3.2 Adjustable-rate mortgages

ARM have both a varying rate and a varying price and the distribution of Figure 4 illustrates the net payments of an annually adjusted 30-year ARM(1) over a holding period of six years.

For an ARM(1) with annually adjustable rates, the price is almost constant and close to par, but as the re-adjustment period increases (in Denmark up to 10 years for ARM(10)) the price may vary considerably as the general level of interest

**FIGURE 4** Distribution of total payments, including interest payments and principal prepayment after six years, of a 30-year ARM(1) with annual adjustments.



rates change similarly to a FRM with maturity of 10 years. In contradistinction to FRM, however, most ARM have no embedded call options and their price might increase to such extremes, which makes it impossible for the mortgagor to prepay the loan, should they decide to quit the loan before the horizon of the fixed-rate term in question.

The interaction between rates and prices, and the uncertainty surrounding the timing for selling the house—due to changing family conditions—makes it difficult to choose an ARM for a particular mortgagor. Nielsen and Poulsen (2004) and Rasmussen and Clausen (2006) proposed models for structuring mortgage loans for homebuyers. However, these models focus on a single product and do not explicitly introduce a risk measure that brings to the surface the diversification issues. A comparison of the distribution of payments for a 30-year ARM(1) with annual adjustments and a 30-year 4% FRM shown in Figure 5, together with the negative correlations of FRM and ARM shown earlier, further highlights the fact that a combination of both types of mortgages should reduce both the average payment and the volatility of payments and in addition impose a limit on the upside potential for high payments in the future.

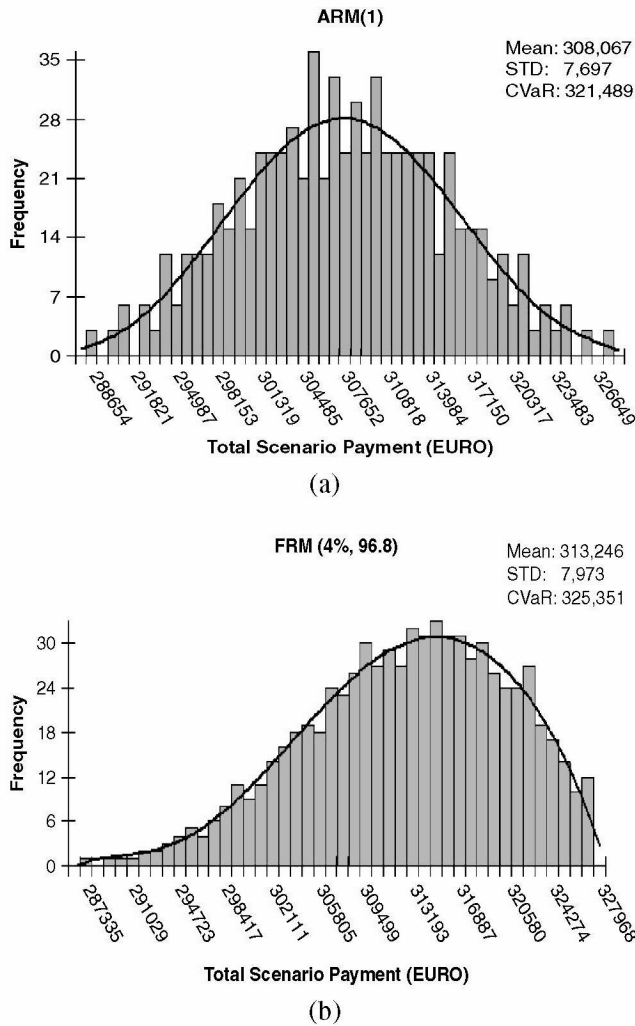
A model for diversifying mortgage loans is introduced next.

#### 4 A DIVERSIFICATION MODEL

The optimization model specifies portfolios that trade-off the net present value of the total mortgage payment against a risk measure of these payments. The risk measure we adopt in this paper is that of CVaR, which has both the theoretical properties of “coherence” (Artzner *et al* 1999) and is also well suited



**FIGURE 5** Comparing the distribution of total payments, including interest payments and principal prepayment after six years, of (a) a 30-year ARM with annual adjustments and (b) a 30-year FRM.



for diversifying portfolios of assets with skewed distributions (Jobst and Zenios 2001).

We are given a set of scenarios  $l \in \Omega$  obtained from the simulation model (see Appendix A) and a set of mortgage loans  $i \in U$ , and the following parameters generated by the simulation model for each scenario:

- $p^l$ , the probability associated with scenario  $l$ ;
- $d^l$ , discount factor under scenario  $l$ ;

- $K_i^l$ , the call price of loan  $i$  under scenario  $l$ ;
- $r_i^l$ , coupon rate for loan  $i$  under scenario  $l$ ;
- $CF_i^l$ , the net present value of payments from one unit of loan  $i$  under scenario  $l$ , including interest and principal payments as well as any fees; and
- $PP_i^l$ , the net present value of prepayments from one unit of loan  $i$  under scenario  $l$  including any retirement of the debt at prevailing market prices.

The following are given input data, relating to features of the problem:

- $IA$ , the initial amount to be borrowed in order to finance the house purchase;
- $P_i$ , price of loan  $i$  at origination time;
- $c$ , variable transaction costs (in percentage); and
- $c_f$ , fixed costs associated with mortgage origination or refinancing.

Finally, we define the model variables:

- $y_i$ , units sold of loan  $i$ ;
- $\zeta$ , value-at-risk (VaR) at the  $100\alpha\%$  confidence level;
- $CVaR(y; \alpha)$ , CVaR of a portfolio with loans  $y = (y_i)_{i \in U}$  at the  $100\alpha\%$  confidence level;
- $y_+^l$ , amount of payment under scenario  $l$  exceeding the VaR level  $\zeta$ ; and
- $Z_i = \begin{cases} 1, & \text{if any amount of loan } i \text{ is originated} \\ 0, & \text{otherwise} \end{cases}$

The optimization model can now be formulated as follows using the linear programming formulation of Rockafellar and Uryasev (2000), where we use  $\lambda$  to denote the degree of risk aversion, ranging from one for high risk aversion to zero for no risk aversion (see also Zenios (2007)):

$$\text{Minimize } (1 - \lambda) \left[ \sum_{i \in U} \sum_{l \in \Omega} p^l (CF_i^l + PP_i^l) y_i \right] + \lambda CVaR(y; \alpha) \quad (1)$$

$$\text{subject to } \sum_{i \in U} P_i y_i \geq IA + \sum_{i \in U} (c y_i + c_f Z_i) \quad (2)$$

$$M Z_i - y_i \geq 0 \quad \text{for all } i \in U \quad (3)$$

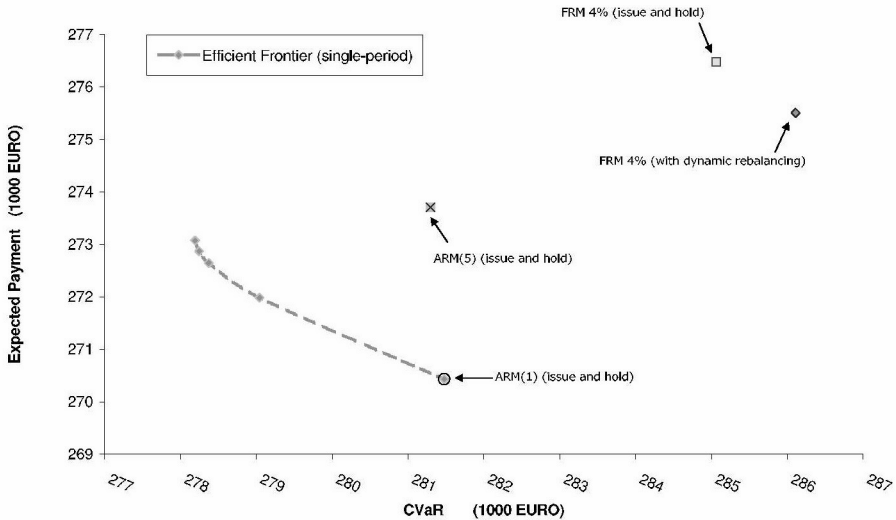
$$y_+^l \geq \left[ \sum_{i \in U} (CF_i^l + PP_i^l) y_i \right] - \zeta \quad \text{for all } l \in \Omega \quad (4)$$

$$CVaR(y; \alpha) = \zeta + \frac{\sum_{l \in \Omega} p^l y_+^l}{1 - \alpha} \quad (5)$$

$$y_i, \zeta, y_+^l \geq 0, \quad Z_i \in \{0, 1\} \quad \text{for all } i \in U, l \in \Omega \quad (6)$$

The objective function (1) trades-off the net present value of total payments (including prepayments) against the risk measure as given by CVaR. Constraint (2) makes sure that we originate enough loans to buy the house at a cost  $IA$  and

**FIGURE 6** The efficient frontier of diversified portfolios of mortgage loans is shown together with performance of individual mortgages in the mean/CVaR space, and the performance of dynamic strategies for rebalancing a single mortgage loan.



pay any transaction costs and the fixed mortgage origination costs. Constraint (3) sets the binary variable  $Z_i$  to one indicating that fixed mortgage origination costs need to be incurred, if any amount of loan  $i$  is chosen in the portfolio of loans, where  $M$  is a large constant to account for the maximum allowable loan.

Constraints (4) and (5) together define the CVaR of the portfolio at the  $100\alpha\%$  confidence level, see Rockafellar and Uryasev (2000) or Zenios (2007). Finally we have the non-negativity constraints (6).

We applied this model to build diversified portfolios of mortgages. First, simulations are employed to develop scenarios of cashflow payments and outstanding principal for both FRM and ARM and then the optimization model is run on the set  $\Omega$ , and for different values of the risk-aversion parameter  $\lambda$ . The results are shown in Figure 6, together with the performance of the individual mortgage loans available to our investor; the benefits from the diversified portfolio become apparent. We observe that the ARM(1) appears as the sole mortgage on the portfolio of only the least risk-averse investors, but as risk aversion increases the portfolios diversify into ARM(5) and FRM as well.

We go a step further, however, and show on the same figure the performance of the loan of a homebuyer who follows a *dynamic strategy* of rebalancing their single FRM as market conditions change. This is clearly a better strategy than issuing and holding a single mortgage throughout and it is, indeed, the strategy pursued by most homebuyers who chose FRM. Even so, we observe from the

results of this figure that the dynamic policy reduces the expected payments, but it does so by assuming higher risks, and it is dominated by the diversified portfolios.

Here, we may rest our case, having demonstrated the validity of the diversification approach for portfolios of mortgage loans. However, an interesting question has been raised that prompts us to further modeling investigations: if a dynamic strategy of rebalancing a single mortgage loan has some advantages over the issue and hold strategy, could it be the case that a dynamic portfolio optimization model would do even better than the model of this section that is defined over a single period, with allowing the possibility of dynamic rebalancing at some future intermediate stages? The answer is affirmative as we see in the next section.

## 5 TAKING A LONG-TERM PERSPECTIVE

The long time horizon of the mortgage decision and the ability of the mortgage owner to rebalance the loan, as market (or family) conditions warrant it, begs for the application of dynamic multi-period portfolio optimization strategies using multi-stage stochastic programming. Such programs have a long history in the optimization literature (see Birge and Louveaux (1997) or Censor and Zenios (1997)) and have been gaining prominence in the risk-management literature since the 1980s (Ziemba (2003); Zenios and Ziemba (2006)). The extension of the model above into a multi-stage setting was developed by Rasmussen and Zenios (2006), where a five-period, four-stage model is developed allowing for refinancing the loan at years one, two, three and five, and maturity at year seven.

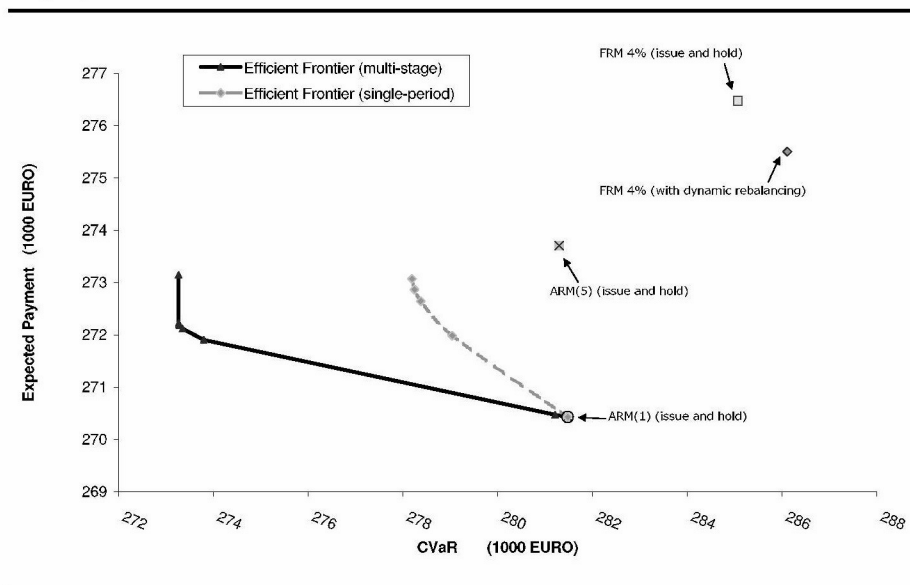
The application of the multi-stage model for optimizing diversified portfolios leads to significant improvements in performance as witnessed from the results of Figure 7, leading to the simultaneous reduction of both the expected net payments and the risk of the payments as measured by CVaR.

The structure of the diversified portfolios obtained with both the single-period and the multi-stage model are shown in Figure 8. We observe that for investors with low risk aversion, ARM are the predominant class of mortgage loans no matter which optimization model is used. However, as risk aversion increases we observe a gradual shift towards the class of FRM and while this trend is common with both models, there is significantly more reliance on FRM for the investor using the multi-stage model. This is so, because with the multi-stage model we can rebalance the portfolio of FRM at the appropriate time for each scenario. The optimal strategy recommended by the multi-stage model is essentially equivalent to synthesizing an ARM with optimal timing for rate re-adjustment at intermediate stages, at one, two, three or five years, depending on the scenario of interest rates. This finding points out that it is worth designing more complex ARM structures that will lock in a rate for pre-specified periods that may depend on the prevailing rates. To do so, however, a legal construct is required so that the synthesized ARM will be presented as a single loan.

## 6 TWO INTERESTING OBSERVATIONS

Finally we use the model to offer answers to two questions that are often raised in the context of mortgage management.

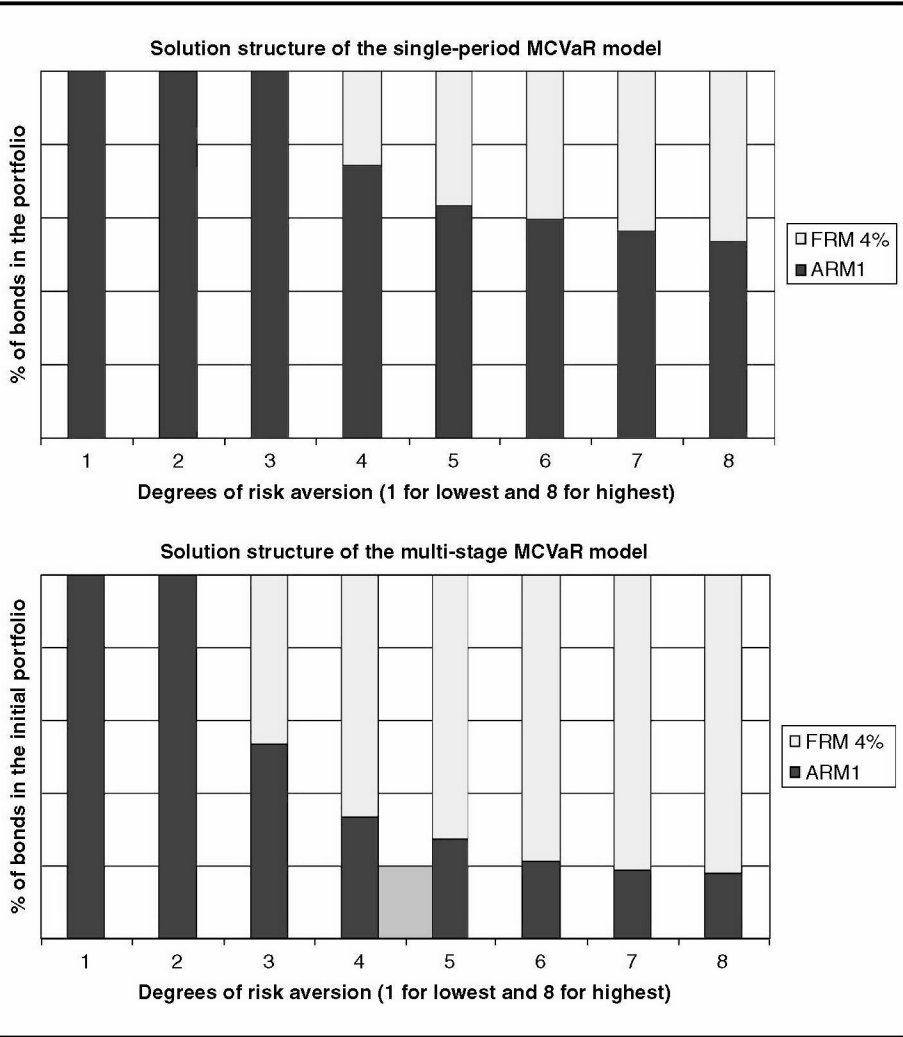
**FIGURE 7** Significant improvements in the performance of diversified portfolios of mortgage loans are realized with the use of a multi-stage model over the single-period model.



## 6.1 The effect of mortgage origination costs

First, we consider the effect of fixed transaction costs for loan origination and the adverse effect this has in rebalancing mortgage loans. Indeed, the arguments against portfolios of loans for mortgages is based on the assumption that the origination costs will be prohibitively high. Mortgage origination costs, in Denmark, include a fee of 1.5% on the required loan paid to the bank upfront for all and any mortgage loans obtained from the bank, a 0.5% penalty for refinancing a loan with a different mortgage and a €300 administration fee for originating every new mortgage. The largest of these fees (1.5% on the loan amount) is a sunk cost, and does not affect our decision to refinance a mortgage, assuming that we stay with the same bank. The rebalancing proportional cost of 0.5% is akin to the transaction costs for any asset management problem and it has been included in all our previous runs. What is left unexamined is the cost for originating new mortgages over and above the original loan. We run the portfolio optimization model with and without the €300 mortgage origination costs. The results are shown in Figure 9. While we note that the performance of the diversified portfolio deteriorates when origination costs are included, it is still the case that they do much better in mean/CVaR space than any of the individual mortgages, and they also outperform the dynamic strategy of rebalancing a single FRM. Hence, loan diversification pays even when accounting for the higher costs of originating multiple loans.

**FIGURE 8** The composition of the diversified portfolio in the aggregate categories of FRM and ARM when using both the single-period and the multi-stage models.

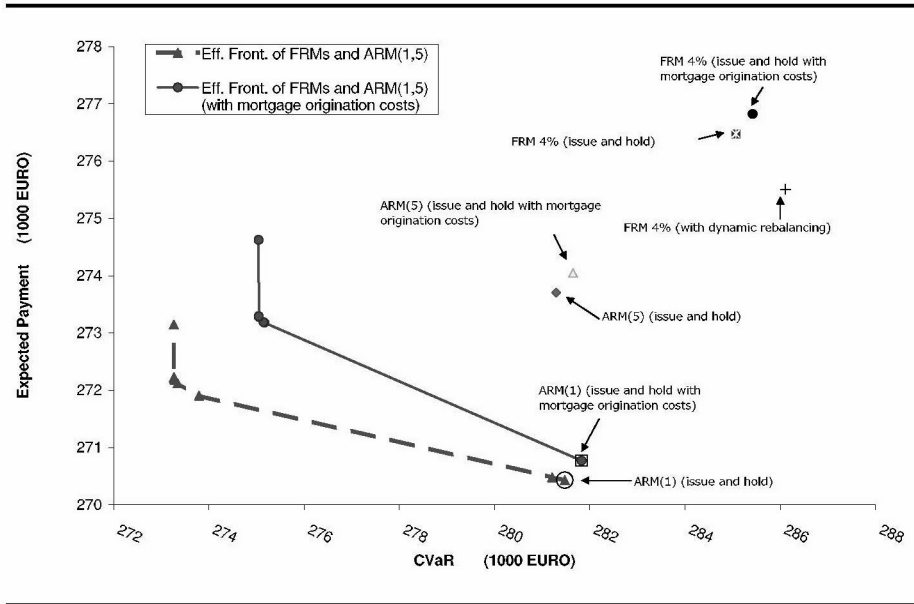


## 6.2 Designing new mortgage products

Armed with the portfolio diversification models we can analyze the effect of new mortgage products on the homebuyers’ portfolios. This has clear implications for the introduction of new products in the market.

Naturally, as we add more products in the market the diversified portfolios will improve in performance or, at least, they will not perform worse as the optimization model will simply ignore any new products that do not contribute to the diversification. Indeed, Figure 10 clearly shows the improvements in efficient frontier as new instruments are added in the universe of mortgage-backed

**FIGURE 9** Efficient frontiers of diversified portfolios of loans with and without the mortgage origination costs.



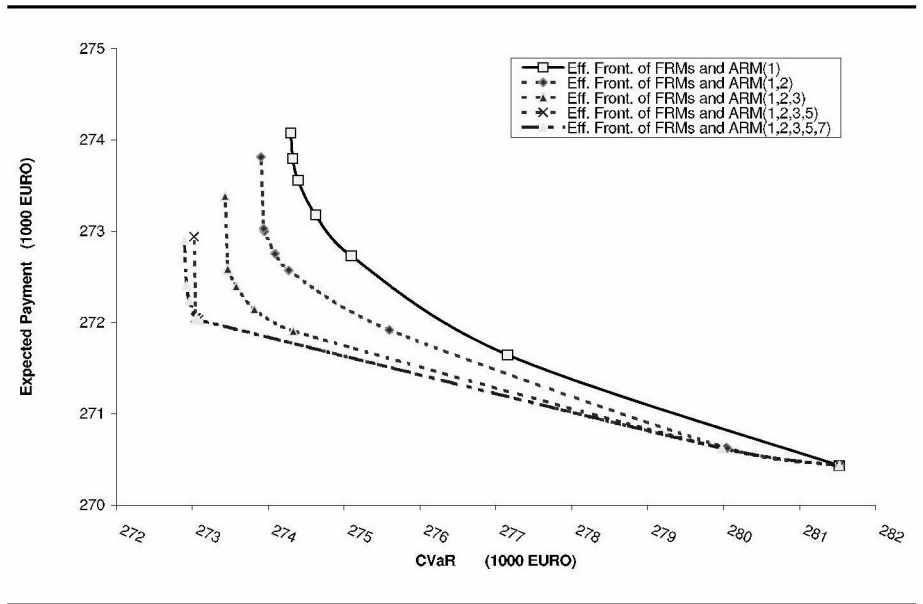
securities, although the improvements are diminishing when adding more than three new securities.

What happens, however, if a bank wishes to issue only one type of  $ARM(t)$  with some to be determined period  $t$  for rate readjustment, to complement a diversified portfolio of FRM and  $ARM(1)$ ? We run the model by introducing, one at a time,  $ARM(2)$ ,  $ARM(5)$  and  $ARM(7)$ . The results are shown in Figure 11, where we observe significant improvements in the performance of the diversified portfolios when we add an  $ARM(2)$  or  $ARM(5)$  to a portfolio with a FRM and  $ARM(1)$ , but things would deteriorate for almost all levels of risk aversion if an  $ARM(7)$  were introduced instead. This analysis provides guidance as to the best mortgage products to be introduced, to maximize the diversification benefits for the homebuyers.

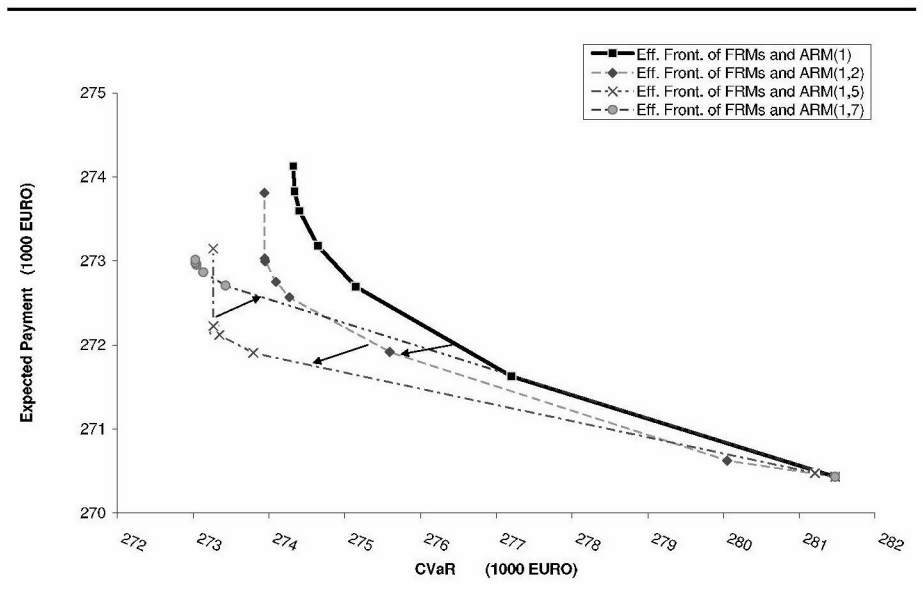
## 7 CONCLUSIONS

We have shown that well-diversified portfolios of mortgage loans can better serve the needs of homebuyers, in both financing the purchase of a home and staying within acceptable risk profiles. This conclusion is robust in the sense that it holds true even in the presence of transaction costs and for short and long horizons alike. From the models developed we have seen that the multi-stage stochastic programming approach is particularly well suited to this type of problem. However, even a single-period model such as the mean/CVaR optimization that has been gaining widespread acceptance in risk management serves the needs of this problem well.

**FIGURE 10** Expanding the universe of mortgage loans available to homebuyers improves the performance of diversified portfolios, although the improvements come at diminishing rates when adding more than three securities.



**FIGURE 11** Designing new mortgage loans: adding new products to the existing portfolio has the optimum positive impact for an ARM(2) and ARM(5) but the diversification effects are diminished for ARM(7).





The models also shed some insights into the introduction of new mortgages in the market.

Finally a word on potential extensions that are possible with the modeling setup we introduced. In this paper we only considered the interest-rate risk. However, the scenario tree can be extended to represent house price and income dynamics in order to capture the wealth risk of the home owner as well.

## APPENDIX A THE SIMULATION MODEL

We use a variation of the Vasicek interest rate model (Jensen and Poulsen (2002)) as the underlying stochastic process to generate estimates of future short rates:

$$dr(t) = \kappa(\theta_P - r(t)) dt + \sigma dW_P(t)$$

where  $r(t)$  is the short rate,  $W$  is a Brownian motion and  $\kappa$ ,  $\theta$  and  $\sigma$  are model parameters controlling the height of the interest rate jumps, the long run mean level of interest rates and the volatility of the interest rates. The model is given under real-world probability measure  $P$ . This can be shifted to the risk-free measure  $Q$  using the transformation:

$$\theta_Q = \theta_P + \pi, \quad \pi \in \mathbb{R}$$

where  $\pi$  is the risk premium, so the Vasicek model under the risk-free probability measure  $Q$  becomes:

$$dr(t) = \kappa(\theta_Q - r(t)) dt + \sigma dW_Q(t)$$

The expected short rates are then found from:

$$E_Q[r(t)] = r_0 \cdot \exp(-\kappa t) + \theta_Q(1 - \exp(-\kappa t))$$

We discretize this short rate process and estimate future rates and prices for mortgage-backed securities using the pricing method of Nielsen and Poulsen (2004).

## REFERENCES

- Artzner, P., Delbaen, F., Eber, J., and Heath, D. (1999). Coherent measures of risk *Mathematical Finance* **9**, 203–228.
- Birge, J. R., and Louveaux, F. (1997). *Introduction to Stochastic Programming*. Springer, Heidelberg.
- Censor, Y., and Zenios, S. A. (1997). *Parallel Optimization: Theory, Algorithms, and Applications (Series on Numerical Mathematics and Scientific Computing)*. Oxford University Press, New York, NY.
- Jensen, B., and Poulsen, R. (2002). Transition densities of diffusion processes: numerical comparison of approximation techniques. *Journal of Derivatives* **9**(4), 18–32.
- Jobst, N. J., and Zenios, S. A. (2001). The tail that wags the dog: integrating credit risk in asset portfolios. *Journal of Risk Finance*, Fall, 31–43.

- Mulvey, J. M., and Zenios, S. A. (1994). Capturing the correlations of fixed-income instruments. *Management Science* **40**, 1329–1342.
- Nielsen, S. S., and Poulsen, R. (2004). A two-factor, stochastic programming model of Danish mortgage-backed securities. *Journal of Economic Dynamics and Control* **28**, 1267–1289.
- Real Estate Center (2002). Could risky mortgage lending practices prick the housing bubble? Web Newsletter, Real Estate Center, Wharton School, University of Pennsylvania.
- Rasmussen, K. M., and Clausen, J. (2006). Mortgage loan portfolio optimization using multi stage stochastic programming. *Journal of Economic Dynamics and Control* **31**, 742–766.
- Rasmussen, K. M., and Zenios, S. A. (2006). Optimal mortgage loan diversification. Available at: [http://www.imm.dtu.dk/Forskning/Soeg\\_publicationer.aspx](http://www.imm.dtu.dk/Forskning/Soeg_publicationer.aspx).
- Rockafellar, R. T., and Uryasev, S. (2000). Optimization of conditional value-at-risk. *The Journal of Risk* **2**(3), 21–41.
- Zenios, S. A. (2007). *Practical Financial Optimization: Decision Making for Financial Engineers*. Blackwell, Oxford.
- Zenios, S. A., and Ziemba, W. T. (2006). *Handbook of Asset and Liability Management. Volume 1: Theory and Methodology (Handbooks in Finance, Vol. 2)*. North-Holland, Amsterdam.
- Ziemba, W. T. (2003). *The Stochastic Programming Approach to Asset, Liability and Wealth Management*. The Research Foundation of AIMR, Association for Investment Management and Research, Charlottesville, VA.